

LOCALLY σ -COMPACT RECTIFIABLE SPACES

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ABSTRACT. A topological space G is said to be a *rectifiable space* provided that there are a surjective homeomorphism $\varphi : G \times G \rightarrow G \times G$ and an element $e \in G$ such that $\pi_1 \circ \varphi = \pi_1$ and for every $x \in G$, $\varphi(x, x) = (x, e)$, where $\pi_1 : G \times G \rightarrow G$ is the projection to the first coordinate. In this paper, we first prove that each locally compact rectifiable space is paracompact, which gives an affirmative answer to Arhangel'skii and Choban's question (Arhangel'skii and Choban [3]). Then we prove that every locally σ -compact rectifiable space with a bc -base is locally compact or zero-dimensional, which improves Arhangel'skii and van Mill's result (Arhangel'skii and van Mill [4]). Finally, we prove that each k_ω -rectifiable space is rectifiable complete.

1. INTRODUCTION

Recall that a *paratopological group* G is a group G with a topology such that the product mapping of $G \times G$ into G is jointly continuous. The space G is called a *topological group* if it is a paratopological group and the inverse mapping of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous. In addition, the space G is said to be a *rectifiable space* provided that there are a surjective homeomorphism $\varphi : G \times G \rightarrow G \times G$ and an element $e \in G$ such that $\pi_1 \circ \varphi = \pi_1$ and for every $x \in G$, $\varphi(x, x) = (x, e)$, where $\pi_1 : G \times G \rightarrow G$ is the projection to the first coordinate. If G is a rectifiable space, then φ is called a *rectification* on G . It is well known that rectifiable spaces and paratopological groups are two good generalizations of topological groups. In fact, for a topological group with the neutral element e , it is easy to see that the mapping $\varphi(x, y) = (x, x^{-1}y)$ is a rectification on G . However, the 7-dimensional sphere S_7 is rectifiable but not a topological group [17, § 3]. The Sorgenfrey line G is a paratopological group with no rectification on G . Further, it is easy to see that each rectifiable space is homogeneous. Recently, the study of rectifiable spaces has become an interesting topic in topological algebra, see [2, 3, 7, 8, 9, 10, 11, 13, 14]. In particular, the following theorem plays an important role in the study of rectifiable spaces.

Theorem 1.1. [5, 7, 16] *A topological space G is rectifiable if and only if there exist an element $e \in G$ and two continuous mappings $p : G^2 \rightarrow G$, $q : G^2 \rightarrow G$ such that for any $x, y \in G$ the next identities hold:*

$$p(x, q(x, y)) = q(x, p(x, y)) = y \text{ and } q(x, x) = e.$$

Given a rectification φ of the rectifiable space G , we can obtain the mappings p and q in Theorem 1.1 as follows. Let $p = \pi_2 \circ \varphi^{-1}$ and $q = \pi_2 \circ \varphi$. Then the mappings p and q satisfy the identities in Theorem 1.1, and both are open mappings.

Let G be a rectifiable space, and let p be the multiplication on G . Then we sometimes write $x \cdot y$ instead of $p(x, y)$ and $A \cdot B$ instead of $p(A, B)$ for any $A, B \subset G$. Obviously, $q(x, y)$ is an element in G such that $x \cdot q(x, y) = y$. Moreover, since $x \cdot e = x \cdot q(x, x) = x$ and $x \cdot q(x, e) = e$,

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it follows that e is a right neutral element for G and $q(x, e)$ is a right inverse for x . Therefore, a rectifiable space G is a topological algebraic system with operations p and q , a 0-ary operation e , and identities as above. Further, the multiplication operation p of this algebraic system need not satisfy the associative law. Note that every topological loop is rectifiable.

In this paper, we mainly discuss the topological properties of locally σ -compact rectifiable spaces. The paper is organized as follows:

In Section 2, we mainly prove that each locally compact rectifiable space is paracompact, which gives an affirmative answer to a question of A.V. Arhangel'skii and M.M. Choban.

In Section 3, we mainly prove that every locally σ -compact rectifiable space with a bc -base is locally compact or zero-dimensional, which improves a result of A.V. Arhangel'skii and J. van Mill.

In Section 4, we mainly prove that each k_ω -rectifiable space is rectifiable complete.

All the spaces considered in this paper are supposed to be Hausdorff unless stated otherwise. The notation ω denotes the first countable infinite order. The letter e denotes the right neutral element of a rectifiable space. For undefined notation and terminologies, the reader may refer to [1] and [6].

2. LOCALLY COMPACT RECTIFIABLE SPACES

In this section, we shall discuss the locally σ -compact rectifiable spaces, and show that each locally σ -compact rectifiable space is paracompact.

In [3], A.V. Arhangel'skii and M.M. Choban posed the following two questions:

Question 2.1. [3, Problem 5.9] *Is every rectifiable p -space paracompact? What if the space is locally compact?*

Question 2.2. [3, Problem 5.10] *Is every rectifiable p -space a D -space?*

It is well known that a paracompact p -space is a D -space. Therefore, if we can prove each rectifiable p -space is paracompact, then the answer to Question 2.2 is also positive. However, we will prove that each locally compact rectifiable space is paracompact, which gives an affirmative answer to the second part of Question 2.1, see Corollary 2.11. By this result, we give a partial answer to Question 2.2, see Corollary 2.14. First of all, we give some concepts and technical lemmas.

A space X is called *locally compact* (resp. *locally σ -compact*) if, for each $x \in X$, there exists a neighborhood U at x in X such that the closure of U in X is compact (resp. σ -compact).

Definition 2.3. [14] Let H be a subspace of a rectifiable space G . Then H is called a *rectifiable subspace* of G if we have $p(H, H) \subset H$ and $q(H, H) \subset H$.

Lemma 2.4. *If A is subset of a rectifiable space G , then $H = \bigcup_{n \in \mathbb{N}} (A_n \cup B_n)$ is the smallest rectifiable subspace containing the set A , where $A_1 = A$, $B_1 = q(A, e) \cup q(A, A)$, $A_{n+1} = p(A_n \cup B_n, A_n \cup B_n)$ and $B_{n+1} = q(A_n \cup B_n, A_n \cup B_n)$ for each $n \in \mathbb{N}$. Moreover, H is open if A is an open subset in G .*

Proof. Since $q(A, A) \subset B_1$, we have $e \in B_1$. Therefore, it is easy to see that $A_n \cup B_n \subset A_{n+1} \cup B_{n+1}$ for each $n \in \mathbb{N}$. Next we shall prove that H is a rectifiable subspace of G . Indeed, take arbitrary points $x, y \in H$. Then there exists an $n \in \mathbb{N}$ such that $x, y \in A_n \cup B_n$, and hence $p(x, y) \in A_{n+1} \cup B_{n+1}$ and $q(x, y) \in A_{n+1} \cup B_{n+1}$. Therefore, H is a rectifiable subspace of G .

Obviously, it follows from the construction of H that H is open if A is an open subset in G . \square

Lemma 2.5. [14] *Let G be a rectifiable space. If H is an open rectifiable subspace of G , then H is closed in G .*

Lemma 2.6. [14] *Let G be a rectifiable space. If H is a rectifiable subspace of G , then the closure of H in G is a rectifiable subspace.*

Proposition 2.7. *Let G be a rectifiable space. If H is a locally compact rectifiable subspace of G , then H is closed in G .*

Proof. Let H_1 be the closure of H in G . Then it follows from Lemma 2.6 that H_1 is a rectifiable subspace of G . Moreover, H is a locally compact subspace of H_1 , which implies that it is open in H_1 . By Lemma 2.5, H is closed in H_1 , hence it is closed in G . \square

Recall that a topological space is κ -Lindelöf, where κ is any cardinal, if every open cover has a subcover of cardinality strictly less than κ . An \aleph_1 -Lindelöf space is called *Lindelöf*. The *Lindelöf degree*, or *Lindelöf number* $l(X)$, is the smallest cardinal κ such that every open cover of the space X has a subcover of size at most κ . Recall that a space is *strongly paracompact* if every open cover has a star-finite open refinement. Obviously, each strongly paracompact space is paracompact.

The following lemma is well known.

Lemma 2.8. *Let X be a compact space and Y be a κ -Lindelöf space. Then $X \times Y$ is a κ -Lindelöf space.*

Theorem 2.9. *If a rectifiable space G contains an open σ -compact rectifiable subspace H , then G admits a disjoint open cover refining the open cover $\{g \cdot H : g \in G\}$. Therefore, G is a topological sum of σ -compact subspaces and hence X is strongly paracompact.*

Proof. Let $l(G) = \gamma$. By induction of cardinals $\alpha \leq \gamma$ we will prove that each rectifiable subspace $X \subset G$ with $H \subset X$ and $l(X) \leq \alpha$ admits a disjoint open cover refining the open cover $\{x \cdot H : x \in X\}$.

Suppose that X is a Lindelöf rectifiable subspace of G such that $H \subset X$. Since X is Lindelöf, there exists a sequence $\{x_n\}_{n \in \omega} \subset X$ such that $\{x_n \cdot H : n \in \omega\}$ is a countable subcover of the open cover $\{x \cdot H : x \in X\}$. For each $n \in \omega$, put

$$U_n = (x_n \cdot H) \setminus \bigcup_{k < n} x_k \cdot H.$$

It follows from Lemma 2.5 that U_n is a clopen subset of G . Therefore, $\mathcal{U} = \{U_n : n \in \omega\}$ is a disjoint open cover of X refining the cover $\{x \cdot H : x \in X\}$.

Now suppose that, for some uncountable cardinal $\kappa \leq \gamma$, we have proved that any rectifiable subspace $X \subset G$ with $H \subset X$ and $l(X) < \kappa$ admits a disjoint open cover refining the cover $\{x \cdot H : x \in X\}$. Next we will show that any rectifiable subspace X with $H \subset X$ and $l(X) = \kappa$ admits a disjoint open cover refining the cover $\{x \cdot H : x \in X\}$.

Fix a rectifiable subspace $X \subset G$ with $H \subset X$ and $l(X) = \kappa$. Since $l(X) = \kappa$, we can take a subset $B \subset X$ such that the cardinality $|B| \leq \kappa$ and $X = \bigcup_{x \in B} x \cdot H$. Enumerate B as $\{x_\alpha : \alpha < \kappa\}$. For each $\alpha < \kappa$, let H_α be the smallest rectifiable subspace of G containing the set $B_\alpha = H \cup \{x_\beta\}_{\beta < \alpha}$. Since $H_\alpha \cdot H \subset H_\alpha$ and H is open, the rectifiable subspace H_α is open in G . It follows from Lemma 2.5 that H_α is closed in G . Since H is σ -compact, it is easy to see that $l(H_\alpha) \leq \alpha < \kappa$ by Lemma 2.8 and the construction of H_α in Lemma 2.4. Hence by the inductive assumption, the rectifiable subspace H_α admits a disjoint open cover \mathcal{U}_α refining the open cover $\{x \cdot H : x \in H_\alpha\}$. Moreover, it easily checked that the union $H^{<\alpha} = \bigcup_{\beta < \alpha} H_\beta$ is an open rectifiable subspace of G . Then the rectifiable subspace $H^{<\alpha}$ is closed in G by Lemma 2.5. Now, for each $\alpha < \kappa$, put

$$\mathcal{V}_\alpha = \{U \setminus H^{<\alpha} : U \in \mathcal{U}_\alpha\}.$$

Obviously, each \mathcal{V}_α is a disjoint open cover of the space $H_\alpha \setminus H^{<\alpha}$ refining the cover \mathcal{U}_α . Put

$$\mathcal{V} = \bigcup_{\alpha < \kappa} \mathcal{V}_\alpha.$$

Then \mathcal{V} is a disjoint open cover of X and refines the open cover $\{x \cdot H : x \in X\}$ of X .

By induction, we can see that G admits a disjoint open cover refining the open cover $\{g \cdot H : g \in G\}$. Since H is open σ -compact, G is a topological sum of σ -compact subspaces, then X is strongly paracompact. \square

Corollary 2.10. *Let G be a locally σ -compact rectifiable space. Then it is a paracompact space.*

Proof. Let U be an open neighborhood of e in G such that the closure of U is σ -compact. Let H be the smallest rectifiable subspace of G containing the set U . It is easy to see that H is open in G , hence it is closed in G by Lemma 2.5. Moreover, it is easy to obtain that H is σ -compact by Lemma 2.4. Then, by Theorem 2.9, G is paracompact. \square

Corollary 2.11. *Let G be a locally compact rectifiable space. Then it is a paracompact space.*

Corollary 2.12. *Each connected locally compact rectifiable space is σ -compact.*

Corollary 2.13. *Each locally compact rectifiable space is normal.*

Corollary 2.14. *Each locally compact rectifiable space is a D -space.*

3. ZERO-DIMENSIONAL RECTIFIABLE SPACES

In this section, we shall discuss rectifiable spaces with a bc -base. We mainly prove that for each rectifiable space with a bc -base G , either G is locally compact or every compact subspace of G is zero-dimensional.

We will call a base \mathcal{B} of a space X a bc -base if the boundary $B(U) = \overline{U} \setminus U$ of every member U of \mathcal{B} is compact.

In [4], A.V. Arhangel'skii and J. van Mill proved the following theorem:

Theorem 3.1. [4] *Every σ -compact non-locally compact topological group with a bc -base is zero-dimensional.*

Next, we will improve this theorem by replacing “topological group” with “rectifiable space”, see Corollary 3.6.

We say that a space X is *separated by a compact subset F of X between points x and y* [4] of X if there are disjoint open subsets U and V such that $x \in U, y \in V$, and $U \cup V = X \setminus F$. A space X is *separated by compacta* [4] if for any two distinct points $x, y \in X$, the space X is separated between x and y by some compact subspace of X .

A topological space X is *totally disconnected* if all connected components in X are the one-point sets.

The following proposition is well known.

Proposition 3.2. *A totally disconnected locally compact space is zero-dimensional.*

Theorem 3.3. *Suppose that G is a rectifiable space such that any two distinct points of G can be separated by a compactum. Then either G is locally compact or every σ -compact subspace of G is zero-dimensional.*

Proof. Suppose that G is not locally compact. By the Countable Closed Sum Theorem of zero-dimensional spaces, it suffices to show that each compact subset of G is zero-dimensional.

Take any compact subset F of G . We will show that F is zero-dimensional.

Claim 1: For any compact subset K of G and any open neighborhood U of e , there exists a point $c \in U$ such that $(F \cdot c) \cap K = \emptyset$.

Assume on the contrary that we have $(F \cdot c) \cap K \neq \emptyset$ for each $c \in U$. Hence there exist points $f \in G$ and $k \in K$ such that $fc = k$, and then

$$c = q(f, p(f, c)) = q(f, k) \in q(F, K).$$

Thus we have $U \subset q(F, K)$. Since q is continuous, $q(F, K)$ is compact, hence G is locally compact, which is a contradiction with our assumption.

Claim 2: For any compact subset K of G , any $x, y \in F$ and any open neighborhoods U and V of x and y in G , respectively, there exists a continuous mapping $h : F \rightarrow G$ such that $h(x) \in U$, $h(y) \in V$ and $K \cap h(F) = \emptyset$.

Indeed, since U and V are open neighborhoods of x and y in G respectively, there exists an open neighborhood W of e in G such that

$$x \cdot W \subset U \text{ and } y \cdot W \subset V.$$

By the Claim 1, there exists a $w \in W$ such that $(F \cdot w) \cap K = \emptyset$. Let $h : F \rightarrow G$ with $f \mapsto f \cdot w$ for each $f \in F$. Obviously, h is continuous. Moreover, we have

$$h(x) = x \cdot w \in x \cdot W \subset U \text{ and } h(y) = y \cdot w \in y \cdot W \subset V.$$

From the Claim 2 and [4, Proposition 3.4], F is totally disconnected. Since F is compact, it follows from Proposition 3.2 that F is zero-dimensional. \square

By Theorem 3.3, the following two corollaries are obvious.

Corollary 3.4. *Suppose that G is a rectifiable space with a bc-base. Then G is locally compact or every compact subspace of G is zero-dimensional.*

Corollary 3.5. *If G is any non-locally compact rectifiable space with a bc-base, then $\text{ind}(G) \leq 1$.*

Corollary 3.6. *Every locally σ -compact rectifiable space with a bc-base is locally compact or zero-dimensional.*

Proof. By Lemma 2.4 and Lemma 2.5, it is easy to see that G has a σ -compact clopen rectifiable subspace. Obviously, it suffices to prove that H is zero-dimensional. Therefore, without loss of generality, we may assume that G is σ -compact. Finally, we only need to apply Corollary 3.4. \square

Moreover, We do not know whether Corollary 3.6 can be extended in the class of paratopological groups. Therefore, we have the following question.

Question 3.7. *Is every locally σ -compact non-locally compact paratopological group with a bc-base zero-dimensional?*

It is well known that a totally disconnected locally compact topological group has an open compact subgroup. Therefore, we have the following question.

Question 3.8. *Does each totally disconnected locally compact rectifiable space have an open compact rectifiable subspace?*

Approximately 10 years ago, L.G. Zambakhidze asked whether every non-zero-dimensional topological group with a bc-base is locally compact. Recently, some progress has been made on this problem in [4]. It is natural to ask the following question.

Question 3.9. *Is every non-zero-dimensional paratopological group (or rectifiable space) with a bc-base locally compact?*

It is well known that a locally compact paratopological group is a topological group, hence we have the following question.

Question 3.10. *Is every non-zero-dimensional paratopological group with a bc-base topological group?*

4. k_ω -RECTIFIABLE SPACES

In [9], Lin defined the concept of a rectifiable complete space and showed that each locally compact rectifiable space is rectifiable complete, and hence each compact rectifiable space is rectifiable complete. In this section, we will prove that each k_ω -rectifiable space G is rectifiable complete. First of all, we recall some concepts and show some technical lemmas.

Definition 4.1. (1) A net $\{x_\alpha\}_{\alpha \in \Gamma}$ in a rectifiable space G is a *left Cauchy net* if for every neighborhood U of e in G there exists an $\alpha_0 \in \Gamma$ such that $q(x_\alpha, x_\beta) \in U$ for every $\alpha, \beta \geq \alpha_0$.

(2) A filter ξ of subsets of a rectifiable space G is said to be a *left Cauchy filter* if for every neighborhood of e in G there exist an $F \in \xi$ and a point $a \in G$ such that $q(a, F) \subset U$.

(3) A rectifiable space G is called *rectifiable complete* if every left Cauchy filter in G converges. Obviously, a rectifiable space G is rectifiable complete if and only if every left Cauchy net in G converges.

(4) Let G be a rectifiable space. If there exists a rectifiable complete space H such that G is dense in H then we say that H is the rectifiable completion of G and denotes it by \widehat{G} .

Definition 4.2. Let G, H be two rectifiable spaces and $f : G \rightarrow H$ be a mapping from G to H . The mapping is called a *homomorphism* if for arbitrary $x, y \in G$ we have $f(p_G(x, y)) = p_H(f(x), f(y))$. Moreover, if f is an one-to-one homomorphism mapping from G onto H , then f is called an *isomorphism*.

In [9], Lin proved that if a rectifiable space G has two rectifiable complete \widehat{G} and \widehat{H} then \widehat{G} and \widehat{H} are topologically isomorphic.

We shall first prove the following lemma which plays an important role in our proof of our main theorem in this section.

Lemma 4.3. *Let F and K be two non-empty compact subsets of a rectifiable space G , and let P be a closed subset of G . If $e \in K \subset F \cdot K$ and $F \cdot K \cap P = \emptyset$, then there exists an open neighborhood U of e such that $F \cdot ((K \cdot U) \cdot U) \cap P = \emptyset$.*

Proof. Claim: For each $y \in K$, there exists an open neighborhood W_y of e such that $F \cdot (y \cdot W_y) \cap P = \emptyset$.

Indeed, fix a point $y \in K$. For each $x \in F$, since $p(x, y) \notin P$, there exists an open neighborhood W_y^x of e such that

$$[(x \cdot W_y^x) \cdot (y \cdot W_y^x)] \cap P = \emptyset.$$

Obviously, each $(x \cdot W_y^x) \cdot (y \cdot W_y^x)$ is an open neighborhood of $x \cdot y$. Then

$$\{x \cdot W_y^x : x \in F\} \text{ and } \{(x \cdot W_y^x) \cdot (y \cdot W_y^x) : x \in F\}$$

are open covers of F and $F \cdot y$, respectively. It follows from the compactness of F and $F \cdot y$, there exists a finite subset C_y of F such that

$$F \subset \bigcup_{x \in C_y} x \cdot W_y^x \text{ and } F \cdot y \subset \bigcup_{x \in C_y} ((x \cdot W_y^x) \cdot (y \cdot W_y^x)).$$

Put $W_y = \bigcap_{x \in C_y} W_y^x$. Then $F \cdot (y \cdot W_y) \cap P = \emptyset$. Obviously, it suffices to prove that

$$z \cdot (y \cdot W_y) \cap P = \emptyset$$

for each $z \in F$. Since $F \subset \bigcup_{x \in C_y} x \cdot W_y^x$, there exists a point $x \in C_y$ such that $z \in x \cdot W_y^x$. Then

$$z \cdot (y \cdot W_y) \subset (x \cdot W_y^x) \cdot (y \cdot W_y) \subset (x \cdot W_y^x) \cdot (y \cdot W_y^x) \subset G \setminus P.$$

Therefore, the Claim holds.

For each $y \in K$, since $p(y, e) = y$, it follows from the Claim that there exist open neighborhoods U_y and V_y of e satisfying the following conditions:

- (1) $U_y \subset V_y$;
- (2) $V_y \subset W_y$;
- (3) $F \cdot (y \cdot W_y) \cap P = \emptyset$;
- (4) $(y \cdot V_y) \cdot V_y \subset y \cdot W_y$;
- (5) $(y \cdot U_y) \cdot U_y \subset y \cdot V_y$.

Then the family $\{F \cdot (y \cdot U_y) : y \in K\}$ and $\{y \cdot U_y : y \in K\}$ are open covers of $F \cdot K$ and K respectively. It follows from the compactness of $F \cdot K$ and K that there exists a finite subset D of K such that

$$F \cdot K \subset \bigcup_{y \in D} F \cdot (y \cdot U_y)$$

and

$$K \subset \bigcup_{y \in D} y \cdot U_y.$$

Put $U = \bigcap_{y \in D} U_y$. Then $F \cdot ((K \cdot U) \cdot U) \cap P = \emptyset$.

Indeed, it suffices to prove that

$$F \cdot ((h \cdot U) \cdot U) \cap P = \emptyset$$

for each $h \in K$. Since $K \subset \bigcup_{y \in D} x \cdot U_y$, there exists a point $y \in D$ such that $h \in y \cdot U_y$. Then

$$\begin{aligned} F \cdot ((h \cdot U) \cdot U) &\subset F \cdot (((y \cdot U_y) \cdot U) \cdot U) \\ &\subset F \cdot (((y \cdot U_y) \cdot U_y) \cdot U) \\ &\subset F \cdot ((y \cdot V_y) \cdot U) \\ &\subset F \cdot ((y \cdot V_y) \cdot V_y) \\ &\subset F \cdot (y \cdot W_y) \\ &\subset G \setminus P. \end{aligned}$$

□

The following lemma was proved in [15].

Lemma 4.4. [15] *Let G be a rectifiable space. If $A \subset G$ and V is an open neighborhood of the right neutral element e of G , then $\overline{A} \subset A \cdot V$.*

We say that the topology of a space X is *determined by a family \mathcal{C}* of its subsets provided that a set $F \subset X$ is closed in X iff $F \cap C$ is closed in C , for each $C \in \mathcal{C}$. If the topology of a space X is determined by an increasing sequence $\{K_n : n \in \omega\}$ of compact subsets in X , then X is called a k_ω -space.

Theorem 4.5. *Let G be a rectifiable space. If G is a k_ω -space, then G is a rectifiable complete space.*

Proof. Assume on the contrary that the rectifiable space G fails to be rectifiable complete. Then G contains a left Cauchy filter ζ of closed subsets with empty intersection. Since G is a k_ω -space, there exists a sequence $\{K_n : n \in \omega\}$ of compact subsets in G such that the topology of G is determined by the family $\{K_n : n \in \omega\}$.

Let

$$L_0 = p(K_0, K_0) \cup q(K_0, K_0)$$

and

$$L_1 = p(K_1, K_1) \cup q(K_1, K_1) \cup p(L_0, L_0) \cup q(L_0, L_0) \cup p(K_1, L_0) \cup q(K_1, L_0).$$

By induction, we can construct a sequence $\{L_n : n \in \omega\}$ of sets in G such that, for each $n \geq 1$, we have

$$L_n = p(K_n, K_n) \cup q(K_n, K_n) \cup p(L_{n-1}, L_{n-1}) \cup q(L_{n-1}, L_{n-1}) \cup p(K_n, L_{n-1}) \cup q(K_n, L_{n-1}).$$

Then we can construct a sequence $\{K_n^* : n \in \omega\}$ of sets in G such that $K_0^* = L_0$ and, for each $n \geq 1$, we have

$$K_n^* = p(L_n, K_{n-1}^*).$$

Moreover, it is easy to see that $p(K_n^*, K_n^*) \subset K_{n+1}^*$ for each $n \in \omega$.

Since each $K_n \subset K_n^*$, the compact sets K_n^* generate the original topology of G , and hence there exists an element $C_n \in \zeta$ such that $K_n^* \cap C_n = \emptyset$. Let us construct a sequence $\{V_i : i \in \omega\}$ of sets in G satisfying the following conditions for each $i \in \omega$:

- (1) $e \in V_i \subset K_i^*$ and V_i is open in K_i^* ;
- (2) $V_i \subset V_{i+1}$;
- (3) $(K_j^* \cdot \overline{V_i}) \cap C_{j+1} = \emptyset$, for each $j \leq i$.

By Lemma 4.3, there exists an open neighborhood U_0 of e in G such that

$$(K_0^* \cdot \overline{U_0}) \cap C_1 = \emptyset.$$

Put $V_0 = U_0 \cap K_0^*$. Assume that we have defined the sets V_0, \dots, V_n satisfying (1)-(3). It follows from (3) that

$$(K_i^* \cdot \overline{V_n}) \cap C_{i+1} = \emptyset$$

for each $i = 0, 1, \dots, n$. Moreover, it is easy to see that each $(K_{n+1}^* \cdot \overline{V_n}) \subset K_{n+2}^*$, so that the choice of the set C_{n+2} implies that

$$(K_{n+1}^* \cdot \overline{V_n}) \cap C_{n+2} = \emptyset.$$

For each $n \in \omega$, since $K_{n+1}^* \cdot \overline{V_n}$ is compact, it follows from Lemma 4.3 that there is an open neighborhood U of e in G such that, for each $i = 0, 1, \dots, n+1$, we have

$$(K_i^* \cdot ((\overline{V_n} \cdot U) \cdot U)) \cap C_{i+1} = \emptyset.$$

Put

$$V_{n+1} = (V_n \cdot U) \cap K_{n+1}^*.$$

By Lemma 4.4, we have

$$\overline{V_{n+1}} \subset V_{n+1} \cdot U \subset (V_n \cdot U) \cdot U,$$

whence it follows that the set V_{n+1} satisfies (3). Obviously, V_{n+1} satisfies (1) and (2).

Put $V = \bigcup_{n=0}^{\infty} V_n$. By (1) and (2), we have

$$V \cap K_n = \bigcup_{i=0}^{\infty} (V_i \cap K_n) = \bigcup_{i=n}^{\infty} (V_i \cap K_n)$$

is open in K_n for each $n \in \omega$. Since the family $\{K_n : n \in \omega\}$ determines the topology of G , we see that V is open in G . From (3) and the definition of V it follows that

$$(K_j \cdot V) \cap C_{j+1} = \emptyset, j \in \omega.$$

Since ζ is a left Cauchy filter in G , we can take a $C \in \zeta$ and $x \in G$ such that $q(x, C) \subset V$, which implies that $C \subset x \cdot V$. Hence there exists an integer $n \geq 1$ such that $x \in K_n$, then $x \cdot V \cap C_{n+1} = \emptyset$. Since $C \subset x \cdot V$, and therefore $C \cap C_{n+1} = \emptyset$, which is a contradiction with $C, C_{n+1} \in \zeta$.

Therefore, G is a rectifiable complete space. \square

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